

Prescribed Ricci curvature on a solid torus

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Abstract

We investigate the prescribed Ricci curvature equation $\text{Ric}(G) = T$ on a solid torus \mathcal{T} under natural boundary conditions. The unknown G here is a Riemannian metric. The letter T in the right-hand side denotes a $(0,2)$ -tensor on \mathcal{T} . We assume T is nondegenerate (in fact, even a lighter assumption would suffice). Our results then settle the questions of the existence and the uniqueness of solutions in the class of rotationally symmetric Riemannian metrics on a neighborhood of the boundary of \mathcal{T} . The paper concludes with a brief discussion of the Einstein equation on \mathcal{T} .

1 Introduction

Our main objective in the present paper is to investigate the prescribed Ricci curvature equation on a solid torus. We focus on the existence and the uniqueness of solutions. It is appropriate to begin with a brief historic review.

Suppose M is a closed manifold. Let T be a $(0,2)$ -tensor on M . Consider the prescribed Ricci curvature equation

$$\text{Ric}(G) = T \tag{1.1}$$

for a Riemannian metric G on M . It is a significant question in geometric analysis whether (1.1) has a solution. In other words, can one find a Riemannian metric G such that its Ricci curvature coincides with T ? The first major step towards answering this question was taken by D. DeTurck. More specifically, assume the tensor T is nondegenerate at a point $x \in M$. Then it is possible to solve (1.1) in a neighborhood of x . This result was originally established in D. DeTurck's paper [10]. Alternative proofs appeared later in various sources including, for example, [4, 15]. Note that the nondegeneracy assumption on T is essential. Without it, the result in [10] would be false. Furthermore, even if T is nondegenerate, one might not be able to solve equation (1.1) on all of M . A strong nonexistence theorem for (1.1) was offered by D. DeTurck and N. Koiso in [11]. It tells us that, whenever T is positive-definite, there is a constant $c_T > 0$ such that $c_T T$ is not the Ricci curvature of any Riemannian metric on M . The reader may see the books [14, 4, 3] for rather detailed surveys of the results discussed in this paragraph. To become familiar with related work, check out [9, 18] and references therein.

One more serious question in geometric analysis is whether a solution of (1.1) is unique in any sense. That is, to what extent does the Ricci curvature determine the Riemannian metric? Progress on this question was made in several papers such as [12, 11, 21, 5, 17] and others. As of today, however, a complete answer is still lacking. The reader may consult [14, 4, 3] for surveys of some of the results. This concludes our historic review. A little more information will be given in the end of Section 3. Meanwhile, we are ready to explain the problem we intend to investigate in the present paper.

Let us update our setup. Suppose M is a compact manifold with boundary ∂M . As before, we assume T is a $(0,2)$ -tensor on M . Consider the prescribed Ricci curvature equation (1.1) on M . New questions arise: Does this equation have a solution satisfying interesting boundary conditions? Is such a solution unique in

any sense? Assume T is nondegenerate. The main result of [10] implies that, given a point x in the interior of M , one can find a neighborhood U of x and a Riemannian metric G on U such that (1.1) holds in U . On the other hand, the nonexistence theorem of [11] suggests that, at least for positive-definite T , it may be problematic to solve (1.1) on all of M ; cf. Remark 3.7 below. Thus, it seems natural to refine the questions we just asked as follows: Given $x \in \partial M$, can one find a neighborhood U of x and a Riemannian metric G such that (1.1) holds in U and G satisfies interesting boundary conditions on $U \cap \partial M$? Is there anything to say about the uniqueness of G ? In other words, to what extent is G determined by its Ricci curvature and its behavior near ∂M ? Up until now, virtually nothing was known about the existence and the uniqueness of solutions to (1.1) on manifolds with boundary. Akin problems were discussed in, for example, [1, 19].

Consider a solid torus \mathcal{T} . The present paper addresses the questions raised in the previous paragraph assuming the manifold M coincides with \mathcal{T} . Related work was done by J. Cao and D. DeTurck in [5]. We will talk about it a little in the end of Section 3. Meanwhile, let us describe our results in more detail. We investigate the prescribed Ricci curvature equation (1.1) near $\partial\mathcal{T}$. The boundary conditions we impose are

$$G_{\partial\mathcal{T}} = R, \quad \text{II}(G) = S. \quad (1.2)$$

Here, $G_{\partial\mathcal{T}}$ is the metric induced by G on $\partial\mathcal{T}$, and $\text{II}(G)$ is the second fundamental form of $\partial\mathcal{T}$ computed in G with respect to the outward unit normal. In the right-hand sides, R and S are $(0, 2)$ -tensors on $\partial\mathcal{T}$. It is clear that R must be positive-definite. Our primary objective in this paper is to investigate the existence and the uniqueness of solutions to equation (1.1) with the boundary conditions (1.2) in the class of rotationally symmetric metrics on a neighborhood of $\partial\mathcal{T}$. It may be appropriate to clarify the terminology here. We say that a Riemannian metric on (a subset of) \mathcal{T} is a rotationally symmetric metric on (that subset of) \mathcal{T} if it satisfies two requirements. Firstly, it is diagonal in the cylindrical coordinates on \mathcal{T} . Secondly, it does not change when pulled back by the standard rotations of \mathcal{T} . Such metrics occur frequently in applications; see, e.g., [8] and references therein.

We can now summarize the main results of the present paper. They appear as Theorems 3.1, 3.4, and 3.8 in the text below. Suppose the tensor T in the right-hand side of (1.1) is nondegenerate (actually, it suffices for our purposes to impose an assumption that is lighter but more difficult to formulate). Theorems 3.1 and 3.4 provide a necessary and sufficient condition for the existence of a rotationally symmetric metric on a neighborhood of $\partial\mathcal{T}$ solving (1.1) and satisfying (1.2). This condition is exceedingly easy to verify. Theorem 3.8 demonstrates that any two rotationally symmetric metrics solving (1.1) in a neighborhood of $\partial\mathcal{T}$ and satisfying (1.2) must be the same in this neighborhood.

We conclude the paper with a quick look at the Einstein equation. Recall that \mathcal{T} is a three-dimensional manifold. Therefore, the solutions of the Einstein equation on \mathcal{T} are metrics with constant sectional curvature. The existence of such metrics on three-dimensional manifolds was the subject of intense research in geometry. The reader may see, e.g., [16] for some contemporary methods and results.

The solutions of the Einstein equation in the class of rotationally symmetric metrics on \mathcal{T} can be found explicitly through a simple computation. It seems appropriate to include the formulas for these solutions in the present paper. They are listed in Proposition 4.1. We believe they will make our exposition more complete. Besides, there is a certain connection between Proposition 4.1 and the topics discussed in, for instance, [1, 7].

N.B.: Throughout the paper, we deal with smooth tensors on (subsets of) \mathcal{T} . This seems to be the most natural way to present the material. It is possible, however, to write down versions of our results for tensors with weaker differentiability properties. We leave it up to the reader to work out the details.

2 Tensors near the boundary of a solid torus

Our principal goal is to study the prescribed Ricci curvature equation on a solid torus. In this section, we set forth the notation, review the necessary background, and state two lemmas. Consider the unit disk $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ in \mathbb{R}^2 . We work with the solid torus $D^2 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$. It will be convenient for us to use the symbol \mathcal{T} for $D^2 \times S^1$. The boundary of \mathcal{T} will be denoted by $\partial\mathcal{T}$. The set $\mathcal{C} = \{(0, 0)\} \times S^1 \subset \mathcal{T}$ is the *core circle* of \mathcal{T} . We employ cylindrical coordinates (λ, μ, r) on \mathcal{T} ; cf., for instance, [8]. The parameters λ and μ run through the interval $(0, 1]$, while the parameter r runs through $[0, 1]$. Recall that \mathcal{T} is a subset of $\mathbb{R}^2 \times \mathbb{R}^2$. The point with the

coordinates (λ, μ, r) in \mathcal{T} coincides with the point $((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that

$$\begin{aligned} x_1 &= r \cos(2\pi\lambda), \quad y_1 = r \sin(2\pi\lambda), \\ x_2 &= \cos(2\pi\mu), \quad y_2 = \sin(2\pi\mu). \end{aligned}$$

Our next step is to discuss two types of rotations of \mathcal{T} . This will help us describe the tensors we will deal with in the sequel.

For every $\lambda_0 \in (0, 1]$, consider the map $\mathcal{V}_{\lambda_0} : \mathcal{T} \rightarrow \mathcal{T}$ defined in cylindrical coordinates by the formula

$$\mathcal{V}_{\lambda_0}((\lambda, \mu, r)) = \begin{cases} (\lambda + \lambda_0, \mu, r), & \text{if } \lambda + \lambda_0 \leq 1, \\ (\lambda + \lambda_0 - 1, \mu, r), & \text{if } \lambda + \lambda_0 > 1. \end{cases}$$

Intuitively, one may view \mathcal{V}_{λ_0} as the rotation of \mathcal{T} by the angle $2\pi\lambda_0$ around the core circle \mathcal{C} . It is easy to see that \mathcal{C} remains fixed under \mathcal{V}_{λ_0} . For each $\mu_0 \in (0, 1]$, consider the map $\mathcal{W}_{\mu_0} : \mathcal{T} \rightarrow \mathcal{T}$ given by the formula

$$\mathcal{W}_{\mu_0}((\lambda, \mu, r)) = \begin{cases} (\lambda, \mu + \mu_0, r), & \text{if } \mu + \mu_0 \leq 1, \\ (\lambda, \mu + \mu_0 - 1, r), & \text{if } \mu + \mu_0 > 1. \end{cases}$$

If one visualizes \mathcal{T} as a “doughnut” in \mathbb{R}^3 , one may think of \mathcal{W}_{μ_0} as the rotation of \mathcal{T} by the angle $2\pi\mu_0$ around the axis passing through the center of \mathcal{C} perpendicular to the plane containing \mathcal{C} . The set of fixed points of \mathcal{W}_{μ_0} is empty unless $\mu_0 = 1$. In this section, we focus on rotationally symmetric tensors on neighborhoods of $\partial\mathcal{T}$. By definition, such tensors possess two properties. Firstly, they are diagonal in the cylindrical coordinates (λ, μ, r) . Secondly, they do not change when pulled back by \mathcal{V}_{λ_0} and \mathcal{W}_{μ_0} for any $\lambda_0 \in (0, 1]$ and $\mu_0 \in (0, 1]$. Rotationally symmetric tensors on neighborhoods of $\partial\mathcal{T}$ admit a simple characterization. Roughly speaking, their components in the coordinates (λ, μ, r) do not depend on the parameters λ and μ . In the next paragraph, we will talk about rotationally symmetric tensors near $\partial\mathcal{T}$ in a more rigorous fashion. But before doing so, we have to introduce one more piece of notation.

Fix $x \in (0, 1]$. Let \mathcal{T}_x stand for the set of points in \mathcal{T} with cylindrical coordinates (λ, μ, r) such that $r \in (1 - x, 1]$. Thus, \mathcal{T}_x is a neighborhood of $\partial\mathcal{T}$ in \mathcal{T} . We say that a smooth (0,2)-tensor T on \mathcal{T}_x is a *rotationally symmetric tensor on \mathcal{T}_x* if it is given in cylindrical coordinates by the formula

$$T = \phi(r) d\lambda \otimes d\lambda + \psi(r) d\mu \otimes d\mu + \sigma(r) dr \otimes dr, \quad r \in (1 - x, 1]. \quad (2.1)$$

In the right-hand side, ϕ , ψ , and σ must be functions from $(1 - x, 1]$ to \mathbb{R} . Granted that T satisfies (2.1), the equalities $\mathcal{V}_{\lambda_0}^* T = T$ and $\mathcal{W}_{\mu_0}^* T = T$ hold for all $\lambda_0 \in (0, 1]$ and $\mu_0 \in (0, 1]$. The asterisks in them designate pullback. Accordingly, a smooth Riemannian metric G on \mathcal{T}_x is a *rotationally symmetric metric on \mathcal{T}_x* if

$$G = f^2(r) d\lambda \otimes d\lambda + g^2(r) d\mu \otimes d\mu + h^2(r) dr \otimes dr, \quad r \in (1 - x, 1]. \quad (2.2)$$

Here, f , g , and h must be functions from $(1 - x, 1]$ to $(0, \infty)$. If G satisfies (2.2), then $\mathcal{V}_{\lambda_0}^* G = G$ and $\mathcal{W}_{\mu_0}^* G = G$ for all $\lambda_0 \in (0, 1]$ and $\mu_0 \in (0, 1]$. Our next step is to state two lemmas of computational nature. We will utilize them in Sections 3 and 4.

Suppose G is a rotationally symmetric metric on \mathcal{T}_x given by (2.2). Its Ricci curvature will be denoted by $\text{Ric}(G)$. The following lemma finds $\text{Ric}(G)$ in terms of f , g , and h . This lemma was provided to us by Andrea Young. The reader may wish to see, e.g., [12, 7, 8] for related computations. In what follows, the subscript r designates the derivative in r .

Lemma 2.1. *The Ricci curvature of the metric G given by (2.2) satisfies the equality*

$$\begin{aligned} \text{Ric}(G) &= \left(-\frac{f f_{rr}}{h^2} + \frac{f f_r h_r}{h^3} - \frac{f f_r g_r}{g h^2} \right) d\lambda \otimes d\lambda, \\ &\quad + \left(-\frac{g g_{rr}}{h^2} + \frac{g g_r h_r}{h^3} - \frac{g f_r g_r}{f h^2} \right) d\mu \otimes d\mu, \\ &\quad + \left(-\frac{f_{rr}}{f} + \frac{f_r h_r}{f h} - \frac{g_{rr}}{g} + \frac{g_r h_r}{g h} \right) dr \otimes dr, \quad r \in (1 - x, 1]. \end{aligned}$$

Proof. Direct calculation. □

The metric G induces a Riemannian metric $G_{\partial\mathcal{T}}$ on $\partial\mathcal{T}$. It is evident that

$$G_{\partial\mathcal{T}} = f^2(1) d\lambda \otimes d\lambda + g^2(1) d\mu \otimes d\mu. \quad (2.3)$$

We denote by $\Pi(G)$ the second fundamental form of $\partial\mathcal{T}$ computed in the metric G with respect to the outward unit normal. It is easy to find $\Pi(G)$ in terms of f , g , and h .

Lemma 2.2. *The equality*

$$\Pi(G) = \frac{f(1)f_r(1)}{h(1)} d\lambda \otimes d\lambda + \frac{g(1)g_r(1)}{h(1)} d\mu \otimes d\mu$$

holds true.

Proof. Another direct calculation. □

3 Prescribed Ricci curvature

This section discusses the existence of solutions to the prescribed Ricci curvature equation in the class of rotationally symmetric metrics on a neighborhood of the boundary of the solid torus \mathcal{T} . We will also address the issue of uniqueness. More precisely, fix $x \in (0, 1]$ and consider a tensor T on \mathcal{T}_x given by formula (2.1). Suppose α , β , η , and θ are real numbers with $\alpha, \beta > 0$. We introduce the (0,2)-tensors R and S on $\partial\mathcal{T}$ by setting

$$\begin{aligned} R &= \alpha^2 d\lambda \otimes d\lambda + \beta^2 d\mu \otimes d\mu, \\ S &= \eta d\lambda \otimes d\lambda + \theta d\mu \otimes d\mu. \end{aligned} \quad (3.1)$$

Our goal is to answer the following questions:

1. Is it possible to find a rotationally symmetric metric G on \mathcal{T}_x such that the Ricci curvature of G equals T , the induced metric $G_{\partial\mathcal{T}}$ equals R , and the second fundamental form $\Pi(G)$ coincides with S ?
2. When such a G exists, is it unique?

In the spirit of [10], we will be imposing a nondegeneracy-type assumption on T . More specifically, we will be requiring that σ stay away from 0. Our first result is a relatively simple necessary condition for the existence of G . It will be demonstrated below that this condition is also sufficient.

Theorem 3.1. *Let G be a rotationally symmetric metric on \mathcal{T}_x such that $\text{Ric}(G) = T$ with the tensor T given by (2.1). Assume $\sigma(1)$ is not equal to 0. Suppose that $G_{\partial\mathcal{T}} = R$ and $\Pi(G) = S$ on $\partial\mathcal{T}$ with the tensors R and S defined by (3.1). Then the quantity*

$$\frac{1}{\sigma(1)} (2\eta\theta + \beta^2\phi(1) + \alpha^2\psi(1))$$

is positive.

Proof. Let us write G in the form (2.2). According to Lemma 2.1, the fact that $\text{Ric}(G) = T$ translates as

$$\begin{aligned} -\frac{ff_{rr}}{h^2} + \frac{ff_r h_r}{h^3} - \frac{ff_r g_r}{gh^2} &= \phi, \\ -\frac{gg_{rr}}{h^2} + \frac{gg_r h_r}{h^3} - \frac{gf_r g_r}{fh^2} &= \psi, \\ -\frac{f_{rr}}{f} + \frac{f_r h_r}{fh} - \frac{g_{rr}}{g} + \frac{g_r h_r}{gh} &= \sigma, \quad r \in (1-x, 1]. \end{aligned} \quad (3.2)$$

We find f_{rr} and g_{rr} from the first two equations and substitute the obtained results in the third equation. This yields

$$\frac{2f_r g_r}{fg} + \frac{h^2}{f^2} \phi + \frac{h^2}{g^2} \psi = \sigma, \quad r \in (1-x, 1].$$

Keeping in mind formula (2.3) and Lemma 2.2, we derive

$$\left(\frac{2\eta\theta h^2(1)}{\alpha^2 \beta^2} + \frac{h^2(1)}{\alpha^2} \phi(1) + \frac{h^2(1)}{\beta^2} \psi(1) \right) = \sigma(1).$$

Consequently,

$$\frac{1}{\sigma(1)} (2\eta\theta + \beta^2 \phi(1) + \alpha^2 \psi(1)) = \frac{\alpha^2 \beta^2}{h^2(1)} > 0.$$

□

Remark 3.2. Suppose the conditions of Theorem 3.1 are satisfied. Then the outward unit normal vector field on $\partial\mathcal{T}$ with respect to the metric G is equal to

$$\frac{1}{h(1)} \frac{\partial}{\partial r} = \sqrt{\frac{2\eta\theta + \beta^2 \phi(1) + \alpha^2 \psi(1)}{\alpha^2 \beta^2 \sigma(1)}} \frac{\partial}{\partial r}.$$

This is a simple consequence of the last formula in the proof above.

Remark 3.3. Like in Theorem 3.1, suppose G is a rotationally symmetric metric on \mathcal{T}_x such that $\text{Ric}(G) = T$, $G_{\partial\mathcal{T}} = R$, and $\text{II}(G) = S$ with T , R , and S given by (2.1) and (3.1). Let us now assume that $\sigma(1) = 0$. Then the equality

$$2\eta\theta + \beta^2 \phi(1) + \alpha^2 \psi(1) = 0$$

holds true. One can easily verify this by retracing the above proof of Theorem 3.1.

We are ready to formulate our next result. It will complete our discussion of the existence of G .

Theorem 3.4. *Let T be a tensor on \mathcal{T}_x given by formula (2.1). Suppose $\sigma(1)$ is not equal to 0. Let R and S be tensors on $\partial\mathcal{T}$ defined by (3.1). Assume that*

$$\frac{1}{\sigma(1)} (2\eta\theta + \beta^2 \phi(1) + \alpha^2 \psi(1)) > 0. \quad (3.3)$$

Then, for some $\epsilon_0 \in (0, x)$, there exists a rotationally symmetric metric G on \mathcal{T}_{ϵ_0} satisfying the equality $\text{Ric}(G) = T$ on \mathcal{T}_{ϵ_0} together with the equalities $G_{\partial\mathcal{T}} = R$ and $\text{II}(G) = S$ on $\partial\mathcal{T}$.

Proof. The argument we use may be viewed as a variant of D. DeTurck's argument from [4, Chapter 5]. We elaborate on this in Remark 3.6 below. Meanwhile, our goal is to find a rotationally symmetric metric G near $\partial\mathcal{T}$ such that $\text{Ric}(G) = T$, $G_{\partial\mathcal{T}} = R$, and $\text{II}(G) = S$. In order to do so, we first consider the system

$$\begin{aligned} \hat{f}_{rr} &= \frac{\hat{h}^2}{\hat{f}} \left(\frac{\hat{f}\hat{f}_r\hat{h}_r}{\hat{h}^3} - \frac{\hat{f}\hat{f}_r\hat{g}_r}{\hat{g}\hat{h}^2} - \phi \circ \hat{h}_r \right), \\ \hat{g}_{rr} &= \frac{\hat{h}^2}{\hat{g}} \left(\frac{\hat{g}\hat{g}_r\hat{h}_r}{\hat{h}^3} - \frac{\hat{g}\hat{f}_r\hat{g}_r}{\hat{f}\hat{h}^2} - \psi \circ \hat{h}_r \right), \\ \hat{h}_{rr} &= \sqrt{\frac{1}{\sigma \circ \hat{h}_r} \left(\frac{2\hat{f}_r\hat{g}_r}{\hat{f}\hat{g}} + \frac{\hat{h}^2}{\hat{f}^2} (\phi \circ \hat{h}_r) + \frac{\hat{h}^2}{\hat{g}^2} (\psi \circ \hat{h}_r) \right)}. \end{aligned} \quad (3.4)$$

Here, the unknown \hat{f} , \hat{g} , and \hat{h} are real-valued functions of the parameter $r \in (1-x, 1]$. We impose the terminal conditions by requiring that

$$\begin{aligned}\hat{f}(1) &= \alpha, \quad \hat{g}(1) = \beta, \quad \hat{h}(1) = 1, \\ \hat{f}_r(1) &= \frac{\eta}{\alpha}, \quad \hat{g}_r(1) = \frac{\theta}{\beta}, \quad \hat{h}_r(1) = 1.\end{aligned}\tag{3.5}$$

These conditions, together with (3.3), ensure that the right-hand sides of equations (3.4) are well-defined when $r = 1$. Further, the expression under the square root symbol is positive then. Employing the Picard-Lindelöf theorem, one can demonstrate that problem (3.4)–(3.5) has a solution on the interval $[1-\epsilon, 1]$ for some $\epsilon \in (0, x)$. More specifically, there exist smooth functions \hat{f} , \hat{g} , and \hat{h} on $[1-\epsilon, 1]$ such that formulas (3.4) hold on $[1-\epsilon, 1]$ and formulas (3.5) hold as well. In particular, these functions do not become 0 on $[1-\epsilon, 1]$. The values of \hat{h}_r on $[1-\epsilon, 1]$ lie in $(1-x, 1]$. Note that \hat{f} , \hat{g} , and \hat{h} are all positive at $r = 1$. The same can be said about the second derivative \hat{h}_{rr} . Therefore, \hat{f} , \hat{g} , and \hat{h} are positive on $[1-\epsilon, 1]$, and we may assume that ϵ is small enough to guarantee that \hat{h}_{rr} is positive on $[1-\epsilon, 1]$.

Our goal is to construct a rotationally symmetric metric G near $\partial\mathcal{T}$ such that $\text{Ric}(G) = T$, $G_{\partial\mathcal{T}} = R$, and $\text{II}(G) = S$. We will utilize the functions \hat{f} , \hat{g} , and \hat{h} obtained in the previous paragraph. Namely, let us introduce the metric \hat{G} on \mathcal{T}_ϵ according to the formula

$$\hat{G} = \hat{f}^2(r) d\lambda \otimes d\lambda + \hat{g}^2(r) d\mu \otimes d\mu + \hat{h}^2(r) dr \otimes dr, \quad r \in (1-\epsilon, 1].$$

Denote $\epsilon_0 = 1 - \hat{h}_r(1-\epsilon)$. Consider the map $\Theta : \mathcal{T}_\epsilon \rightarrow \mathcal{T}_{\epsilon_0}$ given in our cylindrical coordinates by the equality

$$\Theta((\lambda, \mu, r)) = (\lambda, \mu, \hat{h}_r(r)), \quad \lambda \in (0, 1], \quad \mu \in (0, 1], \quad r \in (1-\epsilon, 1].$$

Since \hat{h}_{rr} is positive on $(1-\epsilon, 1]$, this map is a diffeomorphism. We set $G = (\Theta^{-1})^* \hat{G}$, where the asterisk designates pullback. It is clear that G is a rotationally symmetric metric on \mathcal{T}_{ϵ_0} . To complete the proof, we need to show that $\text{Ric}(G) = T$, $G_{\partial\mathcal{T}} = R$, and $\text{II}(G) = S$.

Formulas (3.4) imply

$$\begin{aligned}-\frac{\hat{f}\hat{f}_{rr}}{\hat{h}^2} + \frac{\hat{f}\hat{f}_r\hat{h}_r}{\hat{h}^3} - \frac{\hat{f}\hat{f}_r\hat{g}_r}{\hat{g}\hat{h}^2} &= \phi \circ \hat{h}_r, \\ -\frac{\hat{g}\hat{g}_{rr}}{\hat{h}^2} + \frac{\hat{g}\hat{g}_r\hat{h}_r}{\hat{h}^3} - \frac{\hat{g}\hat{f}_r\hat{g}_r}{\hat{f}\hat{h}^2} &= \psi \circ \hat{h}_r, \\ -\frac{\hat{f}_{rr}}{\hat{f}} + \frac{\hat{f}_r\hat{h}_r}{\hat{f}\hat{h}} - \frac{\hat{g}_{rr}}{\hat{g}} + \frac{\hat{g}_r\hat{h}_r}{\hat{g}\hat{h}} &= \hat{h}_{rr}^2(\sigma \circ \hat{h}_r), \quad r \in (1-\epsilon, 1].\end{aligned}$$

Consequently, in view of Lemma 2.1, the equality $\text{Ric}(\hat{G}) = \Theta^*T$ holds on \mathcal{T}_ϵ . We use this equality to conclude that

$$\text{Ric}(G) = \text{Ric}((\Theta^{-1})^* \hat{G}) = (\Theta^{-1})^* \text{Ric}(\hat{G}) = (\Theta^{-1})^* \Theta^* T = T$$

on \mathcal{T}_{ϵ_0} . It remains to study the behavior of G near the boundary.

Due to conditions (3.5), the metric $\hat{G}_{\partial\mathcal{T}}$ induced by \hat{G} on $\partial\mathcal{T}$ coincides with R . Employing (3.5) and Lemma 2.2, we can also establish that $\text{II}(\hat{G})$ equals S . Besides, the restriction of Θ to $\partial\mathcal{T}$ is the identity map. These facts imply that

$$\begin{aligned}G_{\partial\mathcal{T}} &= (\Theta^{-1})^* \hat{G}_{\partial\mathcal{T}} = \hat{G}_{\partial\mathcal{T}} = R, \\ \text{II}(G) &= \text{II}((\Theta^{-1})^* \hat{G}) = (\Theta^{-1})^* \text{II}(\hat{G}) = \text{II}(\hat{G}) = S.\end{aligned}$$

We have thus verified all the required properties of G . □

Remark 3.5. In the proof of Theorem 3.4, the components of G can be expressed through the components of \hat{G} . More specifically, if G is given by (2.2), one easily sees that

$$f = \hat{f} \circ \hat{h}_r^{-1}, \quad g = \hat{g} \circ \hat{h}_r^{-1}, \quad h = \frac{\hat{h}}{\hat{h}_{rr}} \circ \hat{h}_r^{-1}.\tag{3.6}$$

When carrying out the proof, we derived the equalities $\text{Ric}(G) = T$, $G_{\partial\mathcal{T}} = R$, and $\text{II}(G) = S$ from the formula $G = (\Theta^{-1})^*\hat{G}$. One could also verify them by using Lemmas 2.1 and 2.2 together with (3.6).

Remark 3.6. Chapter 5 of the book [4] describes a method, due to D. DeTurck, of obtaining Riemannian metrics with prescribed Ricci curvature on a neighborhood of an interior point of a manifold. Our proof of Theorem 3.4 may be interpreted as an implementation of that method in the framework of ordinary differential equations.

Remark 3.7. Suppose T' is a smooth positive-definite (0,2)-tensor on \mathcal{T} such that the restriction of T' to \mathcal{T}_x is a rotationally symmetric tensor on \mathcal{T}_x . Assume $\eta, \theta > 0$. Let S be given by the second formula in (3.1). Then, for some $\epsilon'_0 \in (0, x)$, there exists a rotationally symmetric metric G' on $\mathcal{T}_{\epsilon'_0}$ such that $\text{Ric}(G') = T'$ on $\mathcal{T}_{\epsilon'_0}$ and $\text{II}(G') = S$ on $\partial\mathcal{T}$. This is an easy consequence of Theorem 3.4 above. However, there is no smooth Riemannian metric on all of \mathcal{T} such that its Ricci curvature equals T' and the second fundamental form of $\partial\mathcal{T}$ in this metric with respect to the outward unit normal equals S . This follows from Theorem 2 in [20].

The next result establishes the uniqueness of G . Roughly speaking, we will demonstrate that any rotationally symmetric metric \tilde{G} near $\partial\mathcal{T}$ with the same Ricci curvature and the same boundary behavior as G must coincide with G .

Theorem 3.8. *Let T be a tensor on \mathcal{T}_x given by (2.1). Suppose $\sigma(r) \neq 0$ for any $r \in (1-x, 1]$. Let G and \tilde{G} be rotationally symmetric metrics on \mathcal{T}_x satisfying the equality*

$$\text{Ric}(G) = \text{Ric}(\tilde{G}) = T$$

on \mathcal{T}_x together with the equalities $G_{\partial\mathcal{T}} = \tilde{G}_{\partial\mathcal{T}}$ and $\text{II}(G) = \text{II}(\tilde{G})$ on $\partial\mathcal{T}$. Then G coincides with \tilde{G} .

Proof. The argument we use was inspired by an argument proposed by R. Hamilton to establish the uniqueness of solutions to the Ricci flow on a closed manifold. We will say more about this in Remark 3.9 below. Meanwhile, let us consider the set

$$\Omega = \{0\} \cup \{y \in (0, x) \mid G = \tilde{G} \text{ on } \mathcal{T}_y\}$$

and denote $y_0 = \sup \Omega$. The proof of the theorem will be complete if we show that y_0 equals x . Assume this is not the case. Then y_0 must be less than x . We will now use this fact to obtain a contradiction. Our plan is to demonstrate that $G = \tilde{G}$ on $\mathcal{T}_{y_0+\delta_0}$ for some $\delta_0 \in (0, x-y_0)$. This would disagree with the definition of y_0 .

Suppose G has the form (2.2). Consider the ordinary differential equation

$$\hat{h}_{rr} = \frac{\hat{h}}{h \circ \hat{h}_r} \tag{3.7}$$

on the interval $(1-x, 1-y_0]$ for the unknown function \hat{h} . We impose the terminal conditions

$$\hat{h}(1-y_0) = 1, \quad \hat{h}_r(1-y_0) = 1-y_0. \tag{3.8}$$

Employing the Picard-Lindelöf theorem, one can show that problem (3.7)–(3.8) has a unique solution \hat{h} on $[1-y_0-\delta, 1-y_0]$ for some $\delta \in (0, x-y_0)$. The reasoning to be used is the same as the reasoning needed in the proof of Theorem 3.4 to deal with (3.4)–(3.5). The solution \hat{h} is smooth on $[1-y_0-\delta, 1-y_0]$. The values of \hat{h}_r lie in $(1-x, 1]$. Also, \hat{h} and \hat{h}_{rr} are positive. Let us denote $\hat{f} = f \circ \hat{h}_r$ and $\hat{g} = g \circ \hat{h}_r$. These functions will help us show that $G = \tilde{G}$ on an appropriate neighborhood of $\partial\mathcal{T}$.

Set $\delta_1 = 1-y_0-\hat{h}_r(1-y_0-\delta)$. Clearly, $\delta_1 \in (0, x-y_0)$. Our next step is to introduce a map Σ acting from $\mathcal{T}_{y_0+\delta} \setminus \mathcal{T}_{y_0}$ to $\mathcal{T}_{y_0+\delta_1} \setminus \mathcal{T}_{y_0}$ if $y_0 \neq 0$ and from \mathcal{T}_δ to \mathcal{T}_{δ_1} if $y_0 = 0$. Define Σ by the equality

$$\Sigma((\lambda, \mu, r)) = (\lambda, \mu, \hat{h}_r(r)), \quad \lambda \in (0, 1], \quad \mu \in (0, 1], \quad r \in (1-y_0-\delta, 1-y_0],$$

employing our cylindrical coordinates. It is easy to see that Σ is a diffeomorphism. We turn our attention to the metric $\hat{G} = \Sigma^*G$ (the asterisk stands for pullback). Evidently, this metric can be written in the form

$$\hat{G} = \hat{f}^2(r) d\lambda \otimes d\lambda + \hat{g}^2(r) d\mu \otimes d\mu + \hat{h}^2(r) dr \otimes dr, \quad r \in (1-y_0-\delta, 1-y_0].$$

The equation $\text{Ric}(\hat{G}) = \Sigma^* T$ holds true. Together with Lemma 2.1 and the fact that \hat{h}_{rr} must be greater than 0 on $[1 - y_0 - \delta, 1 - y_0]$, this equation implies

$$\begin{aligned}\hat{f}_{rr} &= \frac{\hat{h}^2}{\hat{f}} \left(\frac{\hat{f}\hat{f}_r\hat{h}_r}{\hat{h}^3} - \frac{\hat{f}\hat{f}_r\hat{g}_r}{\hat{g}\hat{h}^2} - \phi \circ \hat{h}_r \right), \\ \hat{g}_{rr} &= \frac{\hat{h}^2}{\hat{g}} \left(\frac{\hat{g}\hat{g}_r\hat{h}_r}{\hat{h}^3} - \frac{\hat{g}\hat{f}_r\hat{g}_r}{\hat{f}\hat{h}^2} - \psi \circ \hat{h}_r \right), \\ \hat{h}_{rr} &= \sqrt{\frac{1}{\sigma \circ \hat{h}_r} \left(\frac{2\hat{f}_r\hat{g}_r}{\hat{f}\hat{g}} + \frac{\hat{h}^2}{\hat{f}^2}(\phi \circ \hat{h}_r) + \frac{\hat{h}^2}{\hat{g}^2}(\psi \circ \hat{h}_r) \right)}, \quad r \in (1 - y_0 - \delta, 1 - y_0].\end{aligned}\quad (3.9)$$

Also, it is not difficult to conclude that

$$\begin{aligned}\hat{f}(1 - y_0) &= f(1 - y_0), \quad \hat{g}(1 - y_0) = g(1 - y_0), \\ \hat{h}(1 - y_0) &= 1, \\ \hat{f}_r(1 - y_0) &= \frac{f_r(1 - y_0)}{h(1 - y_0)}, \quad \hat{g}_r(1 - y_0) = \frac{g_r(1 - y_0)}{h(1 - y_0)}, \\ \hat{h}_r(1 - y_0) &= 1 - y_0.\end{aligned}\quad (3.10)$$

Roughly speaking, the terminal-value problem (3.9)–(3.10) enjoys the uniqueness of solutions. We will use this circumstance to show that $G = \tilde{G}$ on $\mathcal{T}_{y_0 + \delta_0}$ for some $\delta_0 \in (0, x - y_0)$.

Suppose the metric \tilde{G} appears as

$$\tilde{G} = \tilde{f}^2(r) d\lambda \otimes d\lambda + \tilde{g}^2(r) d\mu \otimes d\mu + \tilde{h}^2(r) dr \otimes dr, \quad r \in (1 - x, 1].$$

Here, \tilde{f} , \tilde{g} , and \tilde{h} are positive functions on $(1 - x, 1]$. By analogy with (3.7), let us consider the equation

$$\check{h}_{rr} = \frac{\check{h}}{\tilde{h} \circ \check{h}_r} \quad (3.11)$$

on $(1 - x, 1 - y_0]$ for the unknown \check{h} . Similarly to (3.8), we demand that

$$\check{h}(1 - y_0) = 1, \quad \check{h}_r(1 - y_0) = 1 - y_0. \quad (3.12)$$

Problem (3.11)–(3.12) has a unique solution \check{h} on $[1 - y_0 - \tilde{\delta}, 1 - y_0]$ for some $\tilde{\delta} \in (0, x - y_0)$. Let us introduce the functions $\check{f} = \tilde{f} \circ \check{h}_r$ and $\check{g} = \tilde{g} \circ \check{h}_r$. Arguing as above, we find equations for \check{f}_{rr} , \check{g}_{rr} , and \check{h}_{rr} in terms of \check{f} , \check{g} , \check{h} , \check{f}_r , \check{g}_r , \check{h}_r , ϕ , ψ , and σ . It is easy to understand, in view of the definition of y_0 and the assumptions of the theorem, that

$$\begin{aligned}\check{f}(1 - y_0) &= \tilde{f}(1 - y_0) = f(1 - y_0), \\ \check{g}(1 - y_0) &= \tilde{g}(1 - y_0) = g(1 - y_0), \\ \check{h}(1 - y_0) &= 1, \\ \check{f}_r(1 - y_0) &= \frac{\tilde{f}_r(1 - y_0)}{\tilde{h}(1 - y_0)} = \frac{f_r(1 - y_0)}{h(1 - y_0)}, \\ \check{g}_r(1 - y_0) &= \frac{\tilde{g}_r(1 - y_0)}{\tilde{h}(1 - y_0)} = \frac{g_r(1 - y_0)}{h(1 - y_0)}, \\ \check{h}_r(1 - y_0) &= 1 - y_0.\end{aligned}$$

We conclude that formulas (3.9)–(3.10) will hold on $(1 - y_0 - \tilde{\delta}, 1 - y_0]$ if we replace \hat{f} , \hat{g} , and \hat{h} in them by \check{f} , \check{g} , and \check{h} . This enables us to prove, with the aid of the Picard-Lindelöf theorem, that $\hat{f} = \check{f}$, $\hat{g} = \check{g}$, and

$\hat{h} = \check{h}$ on $[1 - y_0 - \min\{\delta, \tilde{\delta}\}, 1 - y_0]$. Consequently, we have

$$\begin{aligned} f &= \hat{f} \circ \hat{h}_r^{-1} = \check{f} \circ \check{h}_r^{-1} = \tilde{f}, \\ g &= \hat{g} \circ \hat{h}_r^{-1} = \check{g} \circ \check{h}_r^{-1} = \tilde{g}, \\ h &= \frac{\hat{h}}{\hat{h}_{rr}} \circ \hat{h}_r^{-1} = \frac{\check{h}}{\check{h}_{rr}} \circ \check{h}_r^{-1} = \tilde{h}, \quad r \in (1 - y_0 - \delta_0, 1 - y_0], \end{aligned}$$

where the number δ_0 is given by

$$\begin{aligned} \delta_0 &= 1 - y_0 - \hat{h}_r(1 - y_0 - \min\{\delta, \tilde{\delta}\}) \\ &= 1 - y_0 - \check{h}_r(1 - y_0 - \min\{\delta, \tilde{\delta}\}) \in (0, x - y_0). \end{aligned}$$

It becomes obvious that $G = \tilde{G}$ on $\mathcal{T}_{y_0 + \delta_0}$. But this contradicts the definition of y_0 . Hence y_0 equals x , and G coincides with \tilde{G} . \square

Remark 3.9. Our proof of Theorem 3.8 may be interpreted as an adaptation of R. Hamilton's proof from [13] (see also, e.g., [6, Chapter 3]) of the uniqueness of solutions to the Ricci flow on a closed manifold. Let us comment on one aspect of this interpretation. R. Hamilton's proof employed two harmonic map heat flows. The counterparts of these flows in our argument are the ordinary differential equations (3.7) and (3.11).

We mentioned briefly in the introduction that some of the material in [5] was related to the results in the present paper. Let us explain this in more detail. We begin with some notation and terminology. Given a natural number $n \geq 3$, suppose (t, z_1, \dots, z_{n-1}) are hyperspherical coordinates on \mathbb{R}^n . The parameter t here runs through $[0, \infty)$, the parameters z_1, \dots, z_{n-2} through $[0, \pi]$, and the parameter z_{n-1} through $[0, 2\pi)$. The standard Euclidean metric on \mathbb{R}^n appears as

$$dt \otimes dt + \left(\sum_{i=1}^{n-1} H_i^2(t, z_1, \dots, z_{n-1}) dz_i \otimes dz_i \right)$$

away from the origin. In this expression, H_1, \dots, H_{n-1} are functions acting from $(0, \infty) \times [0, \pi]^{n-2} \times [0, 2\pi)$ to $(0, \infty)$. Roughly speaking, a Riemannian metric on \mathbb{R}^n is called a rotationally symmetric metric on \mathbb{R}^n if it can be written as

$$\nu^2(t) dt \otimes dt + v^2(t) \left(\sum_{i=1}^{n-1} H_i^2(1, z_1, \dots, z_{n-1}) dz_i \otimes dz_i \right).$$

In the right-hand side, ν and v must be functions from $(0, \infty)$ to $(0, \infty)$.

The authors of [5] investigated the prescribed Ricci curvature equation on \mathbb{R}^n . In particular, they were concerned with the existence and the uniqueness of solutions in the class of rotationally symmetric metrics on \mathbb{R}^n . The problem came down to a studying a system of two ordinary differential equations with two unknowns. It was possible to simplify the analysis of this system greatly by making a proper change of variables. The proofs of many of the results in [5] exploited this fact.

Consider the closed ball $B_{t_0}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq t_0^2\}$ in \mathbb{R}^n . Here, t_0 is a positive number. The methods of [5] can be used to examine the prescribed Ricci curvature equation on $B_{t_0}^n$, not just on \mathbb{R}^n . With these methods, one may be able to obtain results akin to our Theorems 3.1, 3.4, and 3.8 for metrics near the boundary of $B_{t_0}^n$. This does not seem to be a difficult task. But we will not pursue it.

Our primary goal in the present paper was to investigate the prescribed Ricci curvature equation on \mathcal{T} . We were concerned with the the existence and the uniqueness of solutions among rotationally symmetric metrics on a neighborhood of $\partial\mathcal{T}$. Just like in [5], the problem came down to studying a system of ordinary differential equations; cf. (3.2) above. However, it is not clear whether the analysis of this system can be simplified substantially through a change of variables. Therefore, we do not know if methods similar to those of [5] can be used to prove our Theorems 3.1, 3.4, and 3.8.

4 The Einstein equation

This section briefly discusses the Einstein equation on the solid torus \mathcal{T} . Let us first state a definition. We say that a smooth Riemannian metric G on \mathcal{T} is a *rotationally symmetric metric on \mathcal{T}* if the formula

$$G = f^2(r) d\lambda \otimes d\lambda + g^2(r) d\mu \otimes d\mu + h^2(r) dr \otimes dr, \quad r \in (0, 1], \quad (4.1)$$

holds in our cylindrical coordinates. Here, f , g , and h have to be functions from $(0, 1]$ to $(0, \infty)$. The solutions of the Einstein equation in the class of rotationally symmetric metrics on \mathcal{T} can be found explicitly through a simple computation. We write down these solutions below. Before doing so, however, we need to make a few preparatory remarks.

Let G be a rotationally symmetric metric on \mathcal{T} such that equality (4.1) holds true. It is well known that the function f must then admit a smooth odd extension to $[-1, 1]$. We will denote this extension by the same letter f . Clearly, $f(0)$ is equal to 0. This fact will be essential to our arguments later on. The functions g and h must admit smooth even extensions to $[-1, 1]$. Again, we use the same letters g and h for these extensions. It is clear that $g_r(0)$ and $h_r(0)$ are equal to 0. This will also be important to our arguments later. The derivative $f_r(0)$ must equal $2\pi h(0)$. The values $g(0)$ and $h(0)$ need to be positive. Thus, we have listed several properties of the components of a rotationally symmetric metric on \mathcal{T} in our cylindrical coordinates. Conversely, suppose we have three smooth functions f , g , and h from $[-1, 1]$ to \mathbb{R} . Assume that f is positive on $(0, 1]$ and odd whereas g and h are positive on $[-1, 1]$ and even. If $f_r(0) = 2\pi h(0)$, then equality (4.1) defines a rotationally symmetric metric on \mathcal{T} . The reader may wish to see, e.g., [12, 8] for related material.

In what follows, we fix a number $\tau \in \mathbb{R}$ and set $\kappa = \sqrt{2|\tau|}$. Suppose G is a rotationally symmetric metric on \mathcal{T} satisfying (4.1). The notation $\text{Ric}(G)$ will stand for the Ricci curvature of G . Let us introduce a new coordinate system (λ, μ, s) on \mathcal{T} . We obtain it from (λ, μ, r) by replacing the parameter r with the parameter s connected to r by the formula

$$s(r) = \int_0^r h(\rho) d\rho, \quad r \in [0, 1]. \quad (4.2)$$

The values of s range from 0 to the number s_0 equal to $\int_0^1 h(\rho) d\rho$. In the coordinate system (λ, μ, s) , we have

$$G = \bar{f}^2(s) d\lambda \otimes d\lambda + \bar{g}^2(s) d\mu \otimes d\mu + ds \otimes ds, \quad s \in (0, s_0], \quad (4.3)$$

where \bar{f} and \bar{g} are smooth functions from $[0, s_0]$ to $[0, \infty)$. The comments made in the previous paragraph imply

$$\bar{f}(0) = 0, \quad \bar{f}_s(0) = 2\pi, \quad \bar{g}_s(0) = 0. \quad (4.4)$$

Here and in what follows, the subscript s designates the derivative in s .

Proposition 4.1. *Suppose the rotationally symmetric metric G on \mathcal{T} satisfies equality (4.1) and solves the Einstein equation*

$$\text{Ric}(G) = \tau G \quad (4.5)$$

on \mathcal{T} . If $\tau > 0$, then s_0 lies in $(0, \frac{\pi}{\kappa})$ and there exists a constant $c_1 > 0$ such that

$$G = \frac{8\pi^2}{\kappa^2} (1 - \cos \kappa s) d\lambda \otimes d\lambda + c_1 (1 + \cos \kappa s) d\mu \otimes d\mu + ds \otimes ds, \quad s \in (0, s_0]. \quad (4.6)$$

If $\tau < 0$, then we can find $c_2 > 0$ ensuring that

$$G = \frac{4\pi^2}{\kappa^2} \sinh^2 \kappa s d\lambda \otimes d\lambda + c_2 \cosh^2 \kappa s d\mu \otimes d\mu + ds \otimes ds, \quad s \in (0, s_0]. \quad (4.7)$$

When $\tau = 0$, there is $c_3 > 0$ such that

$$G = 4\pi^2 s^2 d\lambda \otimes d\lambda + c_3 d\mu \otimes d\mu + ds \otimes ds, \quad s \in (0, s_0]. \quad (4.8)$$

Proof. We will only consider the case where $\tau > 0$. Analogous arguments work if $\tau < 0$ or $\tau = 0$. The metric G can be written in the form (4.3). Our goal is to find the functions \bar{f} and \bar{g} . The Einstein equation (4.5) and Lemma 2.1 imply

$$\begin{aligned} -\bar{g}\bar{f}_{ss} - \bar{f}_s\bar{g}_s &= \tau\bar{f}\bar{g}, \\ -\bar{f}\bar{g}_{ss} - \bar{f}_s\bar{g}_s &= \tau\bar{f}\bar{g}, \quad s \in (0, s_0]. \end{aligned} \quad (4.9)$$

Adding these together yields

$$(\bar{f}\bar{g})_{ss} + 2\tau\bar{f}\bar{g} = 0, \quad s \in (0, s_0].$$

In view of (4.4), we conclude that

$$\bar{f}(s)\bar{g}(s) = \frac{2\pi\bar{g}(0)}{\kappa} \sin \kappa s, \quad s \in [0, s_0].$$

It now follows from (4.9) and (4.4) that

$$\begin{aligned} \bar{f}_s(s)\bar{g}(s) &= \pi\bar{g}(0)(\cos \kappa s + 1), \\ \bar{f}(s)\bar{g}_s(s) &= \pi\bar{g}(0)(\cos \kappa s - 1), \quad s \in [0, s_0]. \end{aligned}$$

Manipulating the last three equalities and making use of (4.4) again, we obtain

$$\begin{aligned} \bar{f}(s) &= \frac{2\sqrt{2}\pi \sin \kappa s}{\kappa\sqrt{1 + \cos \kappa s}}, \\ \bar{g}(s) &= \frac{\bar{g}(0)}{\sqrt{2}} \sqrt{1 + \cos \kappa s}, \quad s \in [0, s_0]. \end{aligned}$$

It becomes clear that s_0 must lie in $(0, \frac{\pi}{\kappa})$ and the metric G must satisfy (4.6). \square

Remark 4.2. Proposition 4.1 enables us to make conclusions about the existence of solutions to the Einstein equation on \mathcal{T} with given boundary data. For example, suppose G is a rotationally symmetric metric on \mathcal{T} . Fix two numbers $\alpha, \beta > 0$. Let $G_{\partial\mathcal{T}}$ denote the Riemannian metric on $\partial\mathcal{T}$ induced by G . Proposition 4.1 implies that, if $\tau > 0$ and $\alpha \geq \frac{4\pi}{\kappa}$, equation (4.5) and the equality

$$G_{\partial\mathcal{T}} = \alpha^2 d\lambda \otimes d\lambda + \beta^2 d\mu \otimes d\mu$$

cannot be satisfied simultaneously. This observation is somewhat related to the material in [1, 2].

The converse of Proposition 4.1 holds as well. More precisely, suppose h is a smooth positive even function on $[-1, 1]$. Introduce the coordinate system (λ, μ, s) on \mathcal{T} by replacing the parameter r in the coordinate system (λ, μ, r) with the parameter s related to r through (4.2). If $\tau > 0$ and $s_0 \in (0, \frac{\pi}{\kappa})$, then formula (4.6) defines a rotationally symmetric metric on \mathcal{T} for any $c_1 > 0$. It is not difficult to verify that this metric solves the Einstein equation (4.5). In doing so, Lemma 2.1 may come in handy. Let us now assume $\tau < 0$. Formula (4.7) determines a rotationally symmetric metric on \mathcal{T} for any $c_2 > 0$. One easily checks that this metric satisfies (4.5). Finally, let us assume $\tau = 0$. We arrive to similar conclusions. Namely, (4.8) yields a rotationally symmetric metric on \mathcal{T} for any $c_3 > 0$. This metric solves (4.5).

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